Math 4200-001
Week 3concepts and homework
1.5
Due ~~Wednesday~~ Friday September 11 at start ~~of class~~ 5:00 p.m.

1.5 1ad, 3b, 5c, 6c (in 5c and 6c describe the differential map as a rotation-dilation); 8, 9, 10, 11, 16, 18abc, 19.

w2.1a) Consider $f(z) = \frac{1}{z}$ and $z_0 = \frac{1+i}{2}$. Illustrate the rotation-dilation differential map for f at z_0 using rectangular coordinates. Precisely, Sketch a domain picture containing the point z_0 and coordinate segments through that point having unit tangent vectors 1 and i. Sketch a range picture containing $f(1+i)$, the images of the coordinate segments with the corresponding tangent vectors based at $f(z_0)$ which should be rotated and dilated according to the argument and absolute value of $f'(z_0)$.

w2.1b) Repeat part (a), except using polar form. In other words, for $f(r\,e^{i\,\theta})$, $r_0 = \frac{1}{\sqrt{2}}$, $\theta_0 = \frac{\pi}{4}$, sketch r and θ coordinate segments through z_0 and their tangent vectors. In the range picture sketch the images of these coordinate segments and the corresponding rotated and dilated tangent vectors .

In the problem above you are creating concrete realizations of the schematic pictures Figures 1.5.1 and 1.5.2 in the text.

Math 4200
Wednesday September 9

1.5: From last weeks notes: Review of chain rule for curves and CR equations; precise discussion of the differential map in our context, which is also used more generally for differentiable transformations between Euclidean spaces and differentiable manifolds; new in today's notes: proofs of the full CR theorem and inverse function theorem using Math 3220 ideas.

Announcements: HW due Fri @ 5:00 p.m.
Quiz today is a subs. for w2.1a custom problem
(part b is still due.)

O.H. 1-2 today & Fri

Warmup exercise: Recall the CR equations for the partial derivatives of the real and imaginary parts of a complex differentiable function $f(z)$:

$$f(x+iy) == \underbrace{u(x,y)}_{\mathrm{Re}\, f} + i\, \underbrace{v(x,y)}_{\mathrm{Im}\, f}$$

$$\underline{CR} \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$$

Re-derive the CR equations using the chain rule for curves and the identity

$$f'(z) = f_x = \frac{1}{i} f_y$$

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

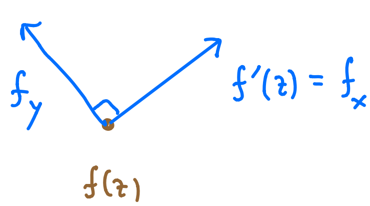
$f = u + iv$

Re dir: $$= \lim_{x+iy_0 \to x_0+iy_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{x - x_0} := f_x(x_0+iy_0) = \boxed{u_x + i v_x} \quad (@\ (x_0,y_0))$$

Im dir: $$\lim_{x_0+iy \to x_0+iy_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{i(y-y_0)} := \frac{1}{i} f_y(x_0+iy) = \boxed{\frac{1}{i}(u_y + i v_y)} \quad "$$

$\frac{1}{i} = -i$

$$CR \begin{cases} \mathrm{Re}: & u_x = v_y \\ \mathrm{Im}: & v_x = -u_y \end{cases}$$



$f_x = \frac{1}{i} f_y$

or $\underbrace{i}_{e^{i\pi/2}} f_x = f_y$: rot. by $\pi/2$ of f_x

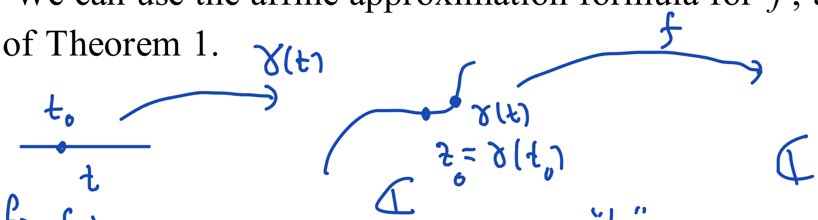
Friday review

Wed

2) Theorem (Chain rule for curves) If f is differentiable at z_0 and $\gamma : I \subseteq \mathbb{R} \to \mathbb{C}$ is a parametric curve $\gamma(t) = x(t) + i\,y(t)$ such that $\gamma(t_0) = z_0$ and such that $\gamma'(t_0) = x'(t_0) + i\,y'(t_0)$ exists, then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0)$$

proof We can use the affine approximation formula for f, at $\gamma(t_0)$, and mimic the proof of Theorem 1.



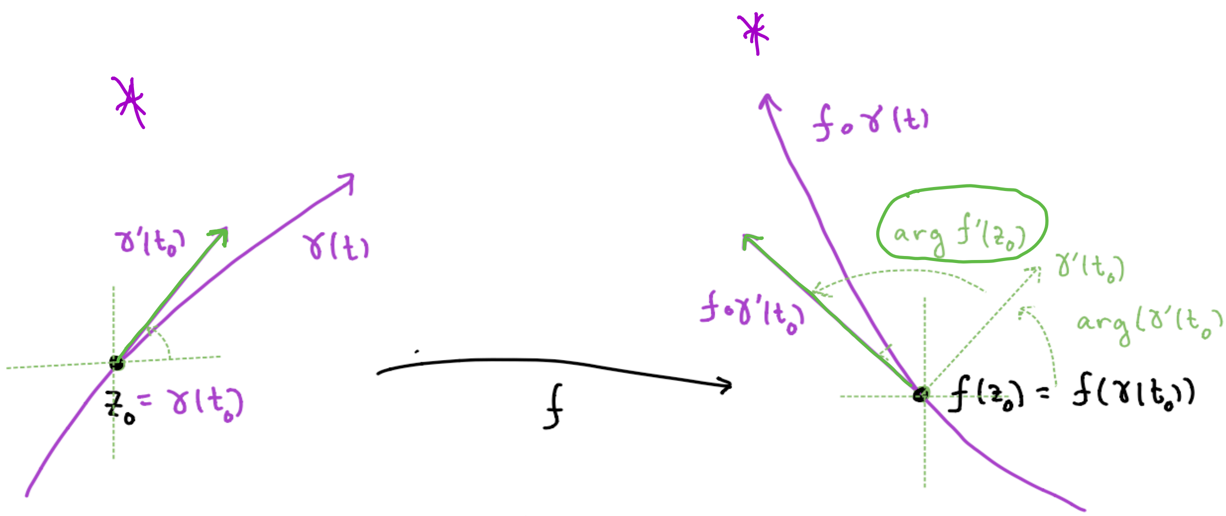
approx for f:

$$f(\gamma(t)) = f(\gamma(t_0)) + f'(\gamma(t_0))\underbrace{(\gamma(t)-\gamma(t_0))}_{\text{"}h\text{"}} + h\,\varepsilon(h) \qquad f \text{ diff'ble @ } z_0$$

$$\frac{f(\gamma(t)) - f(\gamma(t_0))}{t-t_0} = f'(\gamma(t_0))\,\frac{\gamma(t)-\gamma(t_0)}{t-t_0} + \frac{h\,\varepsilon(h)}{t-t_0} \to \gamma'(t_0)\cdot 0 = 0$$

$\lim_{t\to t_0}$: $\boxed{(f\circ\gamma)'(t_0) = f'(\gamma(t_0))\,\gamma'(t_0) + 0}$

Domain-range geometry implied by the chain rule for curves. Consider the curve $\gamma(t)$ which has image in the domain of f, along with the curve $f \circ \gamma(t)$ which has image in the range of f. Let $f'(\gamma(t_0)) = r\,e^{i\theta}$. Then the image curve tangent vector is obtained by rotating the original curve tangent vector by r and scaling it by θ.



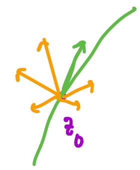
- $(f\circ\gamma)'(t) = \underbrace{f'(\gamma(t))}\gamma'(t)$

- @ t_0 $\quad \underbrace{|f'(z_0)|}_{\text{scaled by mod.}}\ \underbrace{e^{i\arg f'(z_0)}\,\gamma'(t_0)}_{\text{rotated by }\arg f'(z_0)}$

Friday notes

<u>Conformal transformations</u> and <u>differential map</u> discussion:

(i) The precise definition of the *tangent space* at $z_0 \in \mathbb{C}$ is the set of all *tangent vectors* there, i.e. tangent vectors to curves passing through z_0:



$$T_{z_0}\mathbb{C} := \left\{ \gamma'(t_0) \,\middle|\, \gamma \text{ is differentiable at } t_0 \text{ and } \gamma(t_0) = z_0 \right\}$$

$\cong \mathbb{C}$ itself.

(ii) If $f(z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that arises from a real-differentiable function $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, then the *differential of f at z_0* is defined by

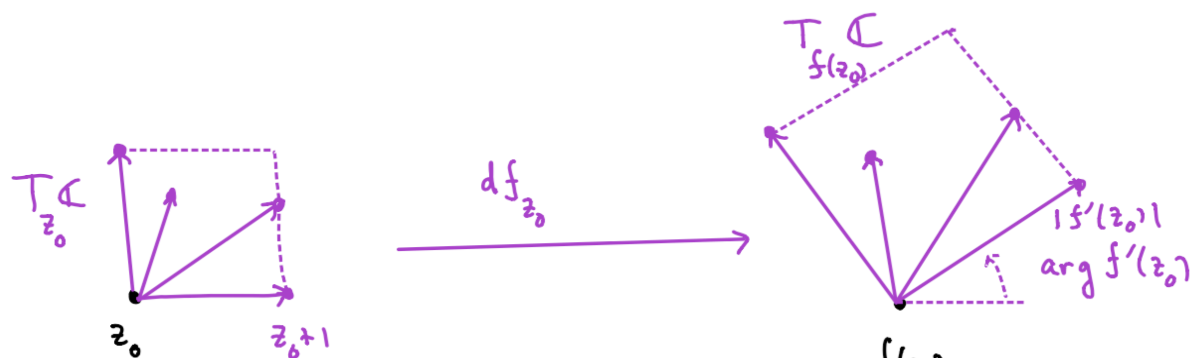
$$df_{z_0}\left(\gamma'(t_0)\right) := (f \circ \gamma)'(t_0).$$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}.$$

(iii) By the chain rule for curves, *if $f(z)$* is complex differentiable at z_0, then

$$df_{z_0}\left(\gamma'(t_0)\right) := (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$$

Geometrically, this means that for complex differentiable functions f, the differential map is a linear transformation from $T_{z_0}\mathbb{C}$ to $T_{f(z_0)}\mathbb{C}$ which is a rotation-dilation.



$$df(\gamma'(t_0)) = f \circ \gamma'(t_0)$$
$$= f'(z_0)\gamma'(t_0)$$

(iv) A function $f: \mathbb{C} \to \mathbb{C}$ is called *conformal* at z_0 iff its differential transformation preserves angles between tangent vectors. Since rotation-dilations have this property, a function f which is complex differentiable at z_0, and for which $f'(z_0) \neq 0$, is conformal at z_0. (It turns out that if f is conformal at z_0 and also preserves orientations of pairs of tangent vectors, then f is complex differentiable at z_0.)

Illustration. Consider

Wed

$f'(z) = 2z$

$$f(z) = z^2, \; z_0 = 1 + i,$$

$$\underbrace{f(z_0)}_{(1+i)^2} = 2i, \; f'(z_0) = 2 + 2i = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$$

Below, are parts of a rectangular coordinate grid in the domain, and the image of that grid in the range space.

because all tangent vects at a pt are rotated & dilated by same amt, ⊥ lines in domain ⟶ ⊥ curves in image

a) Why are the images of the real and imaginary grid lines also perpendicular?

b) Find the formula for the differential map

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$$

and illustrate the rotation dilation.

finish discussion on Wed. next week.

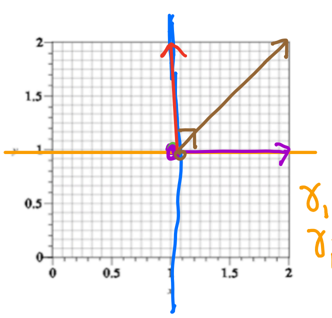
b) $\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\, \gamma'(t)$

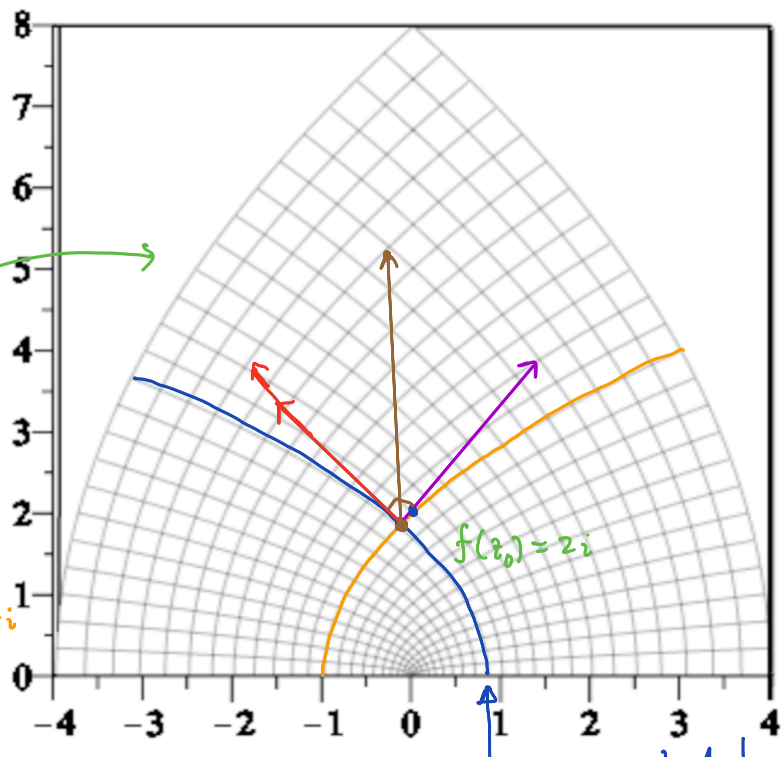
$f'(z_0) = 2\sqrt{2}\, e^{i\pi/4}$

so tangent vectors at z_0 should be rotated by $\pi/4$ & scaled by $2\sqrt{2}$

$f(z) = z^2$

$z_0 = 1 + i$





$\gamma_1(x) = x + i$

$\gamma_1'(1) = 1$

should've hit 1 !

$\gamma_2(y) = 1 + iy$

$\gamma_2'(1) = i$

$f \circ \gamma_1(x) = (x+i)^2$

$= x^2 - 1 + 2ix$

$(f \circ \gamma_1)'(1) = f'(z_0)\, 1$

$= 2\sqrt{2}\, e^{i\pi/4}$

$(f \circ \gamma_2)'(1) = f'(z_0) \underbrace{\gamma_2'(1)}_{i} = 2\sqrt{2}\, e^{i\pi/4}\, i$

The next two theorems are applications of Math 3220 analysis and the correspondence between $f : A \subseteq \mathbb{C} \to \mathbb{C}$, and $F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$: For each

$$f(x + i\,y) = u(x, y) + i\,v(x, y)$$
$$f : A \subseteq \mathbb{C} \to \mathbb{C},\ A \text{ open}$$

there corresponds

$$F(x, y) = (u(x, y), v(x, y))$$
$$F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ A \text{ open}$$

<u>Theorem</u> (full CR Theorem) Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$, $\boldsymbol{z}_0 \in A$. Write

$$f(\boldsymbol{z}) = f(x + i\,y) = u(x, y) + i\,v(x, y)$$

where

$$u(x, y) = \mathrm{Re}(f(x + i\,y),\ v(x, y) = \mathrm{Im}(f(x + i\,y)$$

Then if f is complex differentiable at $\boldsymbol{z}_0 = x_0 + i\,y_0$ *if and only if* the following two conditions hold:

(1) The *Cauchy-Riemann equations* hold at (x_0, y_0):

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0);$$

AND

(2) $F(x, y) = (u(x, y), v(x, y))$ is *Real differentiable* at (x_0, y_0) in the affine approximation sense you discussed in Math 3220. In particular real differentiability is implied by the condition that all of the partial derivatives u_x, y_y, v_x, v_y exist and are continuous in a neigborhood of (x_0, y_0).

HW 18abc

proof:

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$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z_0) = a+bi \text{ exists}$$

$$\Leftrightarrow f(z_0+h) = f(z_0) + (a+bi)h + h\varepsilon(h)$$

$$\text{s.t. } \lim_{h\to 0} \varepsilon(h) = 0$$

write $z = z_0+h$,

$$\Leftrightarrow \lim_{z\to z_0} \frac{|f(z)-f(z_0)-(a+bi)(z-z_0)|}{|z-z_0|} = 0$$

$$\Leftrightarrow \lim_{(x+iy)\to(x_0+iy_0)} \frac{\left| u(x,y)+iv(x,y)-u(x_0)-iv(x_0,y_0) - (a+bi)\ x-x_0+i(y-y_0)) \right|}{|(x-x_0)+i(y-y_0)|} = 0$$

3220

$$F(x,y) = (u(x,y), v(x,y))$$

$$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|} = 0$$

$\Leftrightarrow$ F is real-diff'ble @ (x_0,y_0) with differential matrix @ (x_0,y_0):

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

∎

Theorem (Inverse function theorem) Let f be complex differentiable in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then there exist open sets U, V with $z_0 \in U$, $f(z_0) \in V$ such that $f\colon U \rightarrow V$ is a bijection and $f^{-1}: V \rightarrow U$ is also analytic. Furthermore

$$(f^{-1})'(f(z)) = \frac{1}{f(z)}$$

$\forall z \in U$.

proof:

$\mathbb{C}$	$\mathbb{R}^2$
$f: A \subset \mathbb{C} \rightarrow \mathbb{C}$	$F: A \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

f analytic in a nbhd A of z_0, $f'(z)$ continuous on A, $f'(z_0) \neq 0$,

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

$$f'(z_0) = a + bi = f_x(z_0) = -i f_y(z_0)$$

$$\left(\begin{aligned} a &= u_x(x_0,y_0) = v_y(x_0,y_0) \\ b &= v_x(x_0,y_0) = -u_y(x_0,y_0) \end{aligned} \right)$$

① $\Longrightarrow$

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ real differentiable in a nbhd of (x_0,y_0)

$$DF(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$DF(x_0,y_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ is invertible ($\det = a^2 + b^2 = |f'(z_0)|^2$)

② $\Downarrow$ 3220 inverse fcn thm

$\exists\, \mathcal{U}, V$ open, $(x_0,y_0) \in \mathcal{U} \subset A$; $F(x_0,y_0) \in V$ s.t. $F: \mathcal{U} \rightarrow V$ is a bijection, and $F^{-1}: V \rightarrow \mathcal{U}$ is real differentiable.

Furthermore, by the 3220 chain rule and since

$$F^{-1}(F(x,y)) = (x,y) \quad \forall (x,y) \in \mathcal{U}$$

the matrix product of derivative matrices

$$DF^{-1}(F(x,y))\, DF(x,y) = I \leftarrow \text{identity matrix}$$

In particular,

$$DF^{-1}(F(x,y)) = [DF(x,y)]^{-1} \quad \forall (x,y) \in \mathcal{U}.$$

Note, $DF(x,y) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, so

$$DF^{-1}(F(x,y)) = \frac{1}{\alpha^2+\beta^2} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

is also a rotation-dilation

③ $\Longleftarrow$

$\exists\, \mathcal{U}, V$ open, $z_0 \in A \subset \mathcal{U}$, $f(z_0) \in V$ s.t. $f: \mathcal{U} \rightarrow V$ is a bijection, and $f^{-1}: V \rightarrow \mathcal{U}$ has real differentiable counterpart F^{-1} whose partials satisfy CR eqtns

$\Rightarrow f^{-1}$ is analytic on V.

And since $f^{-1}(f(z)) = z$

$$(f^{-1})'(f(z)) \cdot f'(z) = 1$$

$$(f^{-1})'(f(z)) = \frac{1}{f(z)}, \quad \forall z \in \mathcal{U}.$$

∎

Loose end: (applies to the hw problem 1.5.16)

<u>Theorem</u> Let A be an open connected set in $\mathbb{C}$, $f: A \to \mathbb{C}$ analytic, with $f'(\boldsymbol{z}) = 0\ \forall\ \boldsymbol{z} \in A$. Then f is constant.

proof: For open sets, *connected* and *path-connected are equivalent.* Any continuous path connecting two points in A can be approximated with a continuously differentiable (C^1) path connecting the same two points. Let z_0 be any fixed point in A. Let $z \in A$ be any other point. Let γ be a C^1 curve,

$$\gamma : [a, b] \to A$$
$$\gamma(a) = \boldsymbol{z}_0$$
$$\gamma(b) = \boldsymbol{z}$$

Then by the fundamental theorem of Calculus (applied to the real and imaginary parts of f),

$$\begin{aligned} f(\boldsymbol{z}) - f(\boldsymbol{z}_0) &= \int_a^b \frac{d}{dt} f(\gamma(t))\ \mathrm{d}t \\ &= \int_a^b f'(\gamma(t))\ \gamma'(t)\ \mathrm{d}t \\ &= \int_a^b 0\ \mathrm{d}t\ = 0. \end{aligned}$$

QED

(Or, you showed in Math 3220 that a continuously differentiable function of several variables defined on an open connected set and with all partial derivatives equal to zero, is constant. That theorem applies here, since the partials of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are zero if $f' \equiv 0$.)